

DIFFERENTIATION OF GENUS 3 HYPERELLIPTIC FUNCTIONS

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ABSTRACT. In this work we give an explicit solution to the problem of differentiation of hyperelliptic functions in genus 3 case. It is a genus 3 analogue of the result of F. G. Frobenius and L. Stickelberger [1].

Our method is based on the series of works by V. M. Buchstaber, D. V. Leikin and V. Z. Enolskii [2, 3, 4, 5]. We describe a polynomial map $p: \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$. For $g = 1, 2, 3$ we describe $3g$ polynomial vector fields in \mathbb{C}^{3g} projectable for p and their polynomial Lie algebras. We obtain the corresponding derivations of the field of hyperelliptic functions.

1. INTRODUCTION

In [2] the problem of differentiation of Abelian functions was described. It has deep relations [3] with KdV equations theory.

An *Abelian function* is a meromorphic function on \mathbb{C}^g with a lattice of periods $\Gamma \subset \mathbb{C}^g$ of rank $2g$. We say that an Abelian function is a meromorphic function on the complex torus $T^g = \mathbb{C}^g/\Gamma$.

Let us consider hyperelliptic curves of genus g in the model

$$\mathcal{V}_\lambda = \{(X, Y) \in \mathbb{C}^2: Y^2 = X^{2g+1} + \lambda_4 X^{2g-1} + \lambda_6 X^{2g-2} + \dots + \lambda_{4g} X + \lambda_{4g+2}\}. \quad (1)$$

Such a curve depends on the parameters $\lambda = (\lambda_4, \lambda_6, \dots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2g}$.

Denote by $\mathcal{B} \subset \mathbb{C}^{2g}$ the subspace of parameters such that \mathcal{V}_λ is non-singular for $\lambda \in \mathcal{B}$. We have $\mathcal{B} = \mathbb{C}^{2g} \setminus \Sigma$ where Σ is the discriminant curve.

In this work we will deal only with Abelian functions in the model (1). In [3] such functions are called hyperelliptic.

A hyperelliptic function of genus g is a smooth function defined on an open dense subset of $\mathbb{C}^g \times \mathcal{B}$, such that for each $\lambda \in \mathcal{B}$ it's restriction to $\mathbb{C}^g \times \lambda$ is Abelian with T^g the Jacobian \mathcal{J}_λ of \mathcal{V}_λ .

Let \mathcal{U} be the space of the fiber bundle $\pi: \mathcal{U} \rightarrow \mathcal{B}$ with fiber over $\lambda \in \mathcal{B}$ the Jacobian \mathcal{J}_λ of the curve \mathcal{V}_λ . Thus, a hyperelliptic function is a fiberwise meromorphic function on \mathcal{U} . According to Dubrovin–Novikov theorem [6], the space \mathcal{U} is birationally equivalent to the complex linear space \mathbb{C}^{3g} . Denote the field of hyperelliptic functions by \mathcal{F} .

The general statement of the Problem of Differentiation of Abelian Functions is given in [2]. In this paper we consider the special case of the model (1):

Problem 1.1 (Problem of Differentiation of Hyperelliptic Functions).

- (1) Find the $3g$ generators of the \mathcal{F} -module $\text{Der } \mathcal{F}$ of derivations of the field \mathcal{F} .
- (2) Describe the structure of Lie algebra $\text{Der } \mathcal{F}$ (i.e. find the commutation relations).

In [2] the Problem of Differentiation of Abelian Functions is solved. Yet this solution is not explicit in the sense that it gives general methods that lead to increasing amount of problems and calculations to give explicit answers for each genera.

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In this work we introduce a different approach. The main idea is taken from [3]. It consists of replacing the bundle $\pi : \mathcal{U} \rightarrow \mathcal{B}$ by a polynomial map $p : \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$. This relies on a theorem from [4]. We give polynomial vector fields in \mathbb{C}^{3g} projectable for p and their polynomial Lie algebras [5]. By construction this solves Problem 1.1 for $g = 1, 2, 3$.

The classical genus 1 case formulas are given in [2]. The genus 2 case formulas were first obtained in [3]. This work gives the full explicit answer for Problem 1.1 in genus 3 case. Our method should also work to solve Problem 1.1 for genus $g > 3$.

In this work we use the theory of hyperelliptic Kleinian functions (see [4, 7, 8, 9], and [10] for elliptic functions). Let $\sigma(u, \lambda)$ be the hyperelliptic sigma function (or elliptic sigma function in genus $g = 1$ case). We use the notation

$$\zeta_k = \frac{\partial}{\partial u_k} \ln \sigma(u, \lambda), \quad \wp_{i;k_1, \dots, k_n} = -\frac{\partial^{i+n}}{\partial u_1^i \partial u_{k_1} \dots \partial u_{k_n}} \ln \sigma(u, \lambda), \quad (2)$$

where $u = (u_1, u_3, \dots, u_{2g-1})$, $n \geq 0$, $i + n \geq 2$, $k_s \in \{1, 3, \dots, 2g-1\}$. In the case $n = 0$ we will skip the semicolon. Note that our notation for the variables u_k differs from the one in [4, 9] as $u_i \leftrightarrow u_{2g+1-2i}$. This follows [3] and gives us a supplementary homogeneity for the grading of variables corresponding to their indices.

The paper is organized as follows: In Sections 2–4 we deal with the bundle $\pi : \mathcal{U} \rightarrow \mathcal{B}$. In Section 5 we describe the polynomial map $p : \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$ and its relation to $\pi : \mathcal{U} \rightarrow \mathcal{B}$. In Sections 6–9 we give polynomial vector fields in \mathbb{C}^{3g} projectable for p . In Section 10 we use these vector fields to get the generators of $\text{Der } \mathcal{F}$ and their Lie algebra.

In Section 2 we investigate Problem 1.1, in Section 3 we give its answers [1, 3] for $g = 1, 2$. In Section 4 we give $2g$ polynomial vector fields on \mathcal{B} with some additional properties we will mention.

In Section 6 we relate Problem 1.1 to a problem of constructing polynomial vector fields in \mathbb{C}^{3g} . In Sections 7–9 we give $3g$ polynomial vector fields in \mathbb{C}^{3g} for $g = 1, 2$ and 3, we describe their properties and their polynomial Lie algebras.

In Section 10 we give the theorem solving Problem 1.1 in genus 3 case.

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2. PROBLEM OF DIFFERENTIATION OVER PARAMETERS

Let us consider the Problem of Differentiation of Hyperelliptic Functions 1.1.

In $\mathbb{C}^g \times \mathcal{B} \subset \mathbb{C}^{3g}$ we take the coordinates $(u_1, u_3, \dots, u_{2g-1}, \lambda_4, \lambda_6, \dots, \lambda_{4g}, \lambda_{4g+2})$. Their indices correspond to their grading. Further most functions considered will be homogeneous with respect to this grading. By definition, hyperelliptic functions are meromorphic functions in $u = (u_1, u_3, \dots, u_{2g-1})$ with $2g$ periods that depend on the parameters $\lambda = (\lambda_4, \lambda_6, \dots, \lambda_{4g}, \lambda_{4g+2})$.

If $f(u, \lambda)$ is a hyperelliptic function, then $\frac{\partial}{\partial u_1} f(u, \lambda), \frac{\partial}{\partial u_3} f(u, \lambda), \dots, \frac{\partial}{\partial u_{2g-1}} f(u, \lambda)$, are hyperelliptic functions with the same periods. Therefore, we have $\frac{\partial}{\partial u_k} \in \text{Der } \mathcal{F}$ for $k = 1, 3, \dots, 2g-1$. We denote $\frac{\partial}{\partial u_k} = \mathcal{L}_k$ for odd k from 1 to $2g-1$.

On the other hand, if we take a hyperelliptic function as a smooth function defined on an open dense subset of $\mathbb{C}^g \times \mathcal{B}$, its derivative $\frac{\partial}{\partial \lambda_k} f(u, \lambda)$ in general would not be hyperelliptic, thus $\frac{\partial}{\partial \lambda_k} \notin \text{Der } \mathcal{F}$. Therefore, Problem 1.1 has the following subproblem:

Problem 2.1. *For a vector field L in \mathcal{B} find a vector field \mathcal{L} in $\mathbb{C}^g \times \mathcal{B}$ such that $\mathcal{L} \in \text{Der } \mathcal{F}$, the vector field \mathcal{L} is projectable for π and L is the pushforward of \mathcal{L} (i.e. $\mathcal{L}(\pi^*(f)) = \pi^*(L(f))$ for any function f in \mathcal{B}).*

A solution to Problem 2.1 for $2g$ vector fields L_{2k} independent at any point of \mathcal{B} together with the vector fields \mathcal{L}_k for $k = 1, 3, \dots, 2g-1$ would give a solution to Problem 1.1 (1).

3. KNOWN SOLUTIONS OF THE PROBLEM OF DIFFERENTIATION OF HYPERELLIPTIC FUNCTIONS

3.1. Genus 1. In the elliptic case the generators of the \mathcal{F} -module $\text{Der } \mathcal{F}$ have been found in [1]. We give the generators and their Lie algebra:

$$\begin{aligned}\mathcal{L}_0 &= 4\lambda_4\partial_{\lambda_4} + 6\lambda_6\partial_{\lambda_6} - u_1\partial_{u_1}, & [\mathcal{L}_0, \mathcal{L}_1] &= \mathcal{L}_1, \\ \mathcal{L}_1 &= \partial_{u_1}, & [\mathcal{L}_0, \mathcal{L}_2] &= 2\mathcal{L}_2, \\ \mathcal{L}_2 &= 6\lambda_6\partial_{\lambda_4} - \frac{4}{3}\lambda_4^2\partial_{\lambda_6} - \zeta_1\partial_{u_1}, & [\mathcal{L}_1, \mathcal{L}_2] &= \wp_2\mathcal{L}_1.\end{aligned}$$

3.2. Genus 2. In this case Problem 1.1 was solved in [3]. The generators are (see [3], Theorem 29):

$$\begin{aligned}\mathcal{L}_0 &= L_0 - u_1\partial_{u_1} - 3u_3\partial_{u_3}, & \mathcal{L}_2 &= L_2 + \left(-\zeta_1 + \frac{4}{5}\lambda_4u_3\right)\partial_{u_1} - u_1\partial_{u_3}, \\ \mathcal{L}_1 &= \partial_{u_1}, & \mathcal{L}_4 &= L_4 + \left(-\zeta_3 + \frac{6}{5}\lambda_6u_3\right)\partial_{u_1} - (\zeta_1 + \lambda_4u_3)\partial_{u_3}, \\ \mathcal{L}_3 &= \partial_{u_3}, & \mathcal{L}_6 &= L_6 + \frac{3}{5}\lambda_8u_3\partial_{u_1} - \zeta_3\partial_{u_3},\end{aligned}$$

where the vector fields L_k on \mathcal{B} will be given explicitly in Section 4.

The Lie algebra is (see [3], Theorem 32):

$$\begin{aligned}[\mathcal{L}_0, \mathcal{L}_k] &= k\mathcal{L}_k, \quad k = 1, 2, 3, 4, 6, & [\mathcal{L}_1, \mathcal{L}_3] &= 0, \\ [\mathcal{L}_1, \mathcal{L}_2] &= \wp_2\mathcal{L}_1 - \mathcal{L}_3, & [\mathcal{L}_1, \mathcal{L}_4] &= \wp_{1;3}\mathcal{L}_1 + \wp_2\mathcal{L}_3, & [\mathcal{L}_1, \mathcal{L}_6] &= \wp_{1;3}\mathcal{L}_3 \\ [\mathcal{L}_3, \mathcal{L}_2] &= \left(\wp_{1;3} + \frac{4}{5}\lambda_4\right)\mathcal{L}_1, & [\mathcal{L}_3, \mathcal{L}_6] &= \frac{3}{5}\lambda_8\mathcal{L}_1 + \wp_{0;3,3}\mathcal{L}_3, \\ [\mathcal{L}_3, \mathcal{L}_4] &= \left(\wp_{0;3,3} + \frac{6}{5}\lambda_6\right)\mathcal{L}_1 + (\wp_{1;3} - \lambda_4)\mathcal{L}_3, \\ [\mathcal{L}_2, \mathcal{L}_4] &= \frac{8}{5}\lambda_6\mathcal{L}_0 - \frac{8}{5}\lambda_4\mathcal{L}_2 + 2\mathcal{L}_6 - \frac{1}{2}\wp_{2;3}\mathcal{L}_1 + \frac{1}{2}\wp_3\mathcal{L}_3, \\ [\mathcal{L}_2, \mathcal{L}_6] &= \frac{4}{5}\lambda_8\mathcal{L}_0 - \frac{4}{5}\lambda_4\mathcal{L}_4 - \frac{1}{2}\wp_{1;3,3}\mathcal{L}_1 + \frac{1}{2}\wp_{2;3}\mathcal{L}_3, \\ [\mathcal{L}_4, \mathcal{L}_6] &= -2\lambda_{10}\mathcal{L}_0 + \frac{6}{5}\lambda_8\mathcal{L}_2 - \frac{6}{5}\lambda_6\mathcal{L}_4 + 2\lambda_4\mathcal{L}_6 - \frac{1}{2}\wp_{0;3,3,3}\mathcal{L}_1 + \frac{1}{2}\wp_{1;3,3}\mathcal{L}_3.\end{aligned}$$

4. POLYNOMIAL VECTOR FIELDS IN \mathcal{B}

We consider \mathbb{C}^{2g} with coordinates $(\lambda_4, \lambda_6, \dots, \lambda_{4g}, \lambda_{4g+2})$ and set $\lambda_s = 0$ for every $s \notin \{4, 6, \dots, 4g, 4g+2\}$. For $k, m \in \{1, 2, \dots, 2g\}$, $k \leq m$ set

$$T_{2k,2m} = 2(k+m)\lambda_{2k+2m} + \sum_{s=2}^{k-1} 2(k+m-2s)\lambda_{2s}\lambda_{2k+2m-2s} - \frac{2k(2g-m+1)}{2g+1}\lambda_{2k}\lambda_{2m},$$

and for $k > m$ set $T_{2k,2m} = T_{2m,2k}$. For $k = 0, 1, 2, \dots, 2g-1$ we have the vector fields

$$L_{2k} = \sum_{s=2}^{2g+1} T_{2k+2,2s-2} \frac{\partial}{\partial \lambda_{2s}}. \tag{3}$$

Properties 4.1.

- (1) The vector field L_0 is the Euler vector field on \mathcal{B} .
We have $L_0(\lambda_k) = k\lambda_k$, $[L_0, L_{2k}] = 2kL_{2k}$.
- (2) We have $L_{2k}(\lambda_{2s+4}) = L_{2s}(\lambda_{2k+4})$.
- (3) The vector fields L_{2k} are independent at any point of \mathcal{B} .
- (4) The vector fields L_{2k} are tangent to the discriminant curve Σ of \mathcal{V}_λ .

Let us give some remarks. Note that properties (1) and (2) follow directly from definition. The property (3) is the one we will use essentially further in the work. For the property (4) let us introduce the following notation. Consider the curve \mathcal{V}_λ . Set

$$f(X) = X^{2g+1} + \lambda_4 X^{2g-1} + \lambda_6 X^{2g-2} + \dots + \lambda_{4g} X + \lambda_{4g+2}.$$

Denote by $R(\lambda)$ the resultant of $f(X)$ and $f'(X)$. The discriminant curve Σ is defined by $(\lambda \in \Sigma) \Leftrightarrow (R(\lambda) = 0)$. Thus, property (4) is $L_{2k}(R(\lambda)) = 0$ for λ such that $R(\lambda) = 0$.

We will prove the Properties 4.1 for the vector fields (3) for $g = 1, 2, 3$. Let us note that for general g various constructions have been given to solve the problem of finding vector fields of the form (3) with Properties 4.1 (see [11], [12] section 4). We will need the explicit form (3) for these vector fields.

Lemma 4.2. *The vector fields L_{2k} , $k = 0, 1, \dots, 2g-1$, have properties 4.1 for $g = 1, 2, 3$.*

Proof. Denote by T the symmetric $2g \times 2g$ matrix with elements $T_{2k, 2m} = L_{2k-2}(\lambda_{2m+2})$.

For $g = 1$ we have

$$T = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 \\ 6\lambda_6 & -\frac{4}{3}\lambda_4^2 \end{pmatrix}, \quad \det T = -\frac{4}{3}(4\lambda_4^3 + 27\lambda_6^2), \quad R(\lambda) = 4\lambda_4^3 + 27\lambda_6^2. \quad (4)$$

We have $(R(\lambda) = 0) \Leftrightarrow (\det T = 0)$, therefore the vector fields L_0 and L_2 are independent at any point of $\mathcal{B} = \mathbb{C}^2 \setminus \Sigma$. We have $L_0 \det T = 12 \det T$, $L_2 \det T = 0$, so the vector fields L_0 and L_2 are tangent to Σ .

For $g = 2$ we have

$$T = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} & 0 \\ 8\lambda_8 & 10\lambda_{10} & 4\lambda_4\lambda_8 & 6\lambda_4\lambda_{10} \\ 10\lambda_{10} & 0 & 6\lambda_4\lambda_{10} & 4\lambda_6\lambda_{10} \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 12\lambda_4^2 & 8\lambda_4\lambda_6 & 4\lambda_4\lambda_8 \\ 0 & 8\lambda_4\lambda_6 & 12\lambda_6^2 & 6\lambda_6\lambda_8 \\ 0 & 4\lambda_4\lambda_8 & 6\lambda_6\lambda_8 & 8\lambda_8^2 \end{pmatrix}. \quad (5)$$

This coincides with the matrix given in [2], example 15, so we give this reference for the proof. We have as well $(R(\lambda) = 0) \Leftrightarrow (\det T = 0)$.

For $g = 3$ we have

$$T = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} & 12\lambda_{12} & 14\lambda_{14} \\ 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} & 12\lambda_{12} & 14\lambda_{14} & 0 \\ 8\lambda_8 & 10\lambda_{10} & 12\lambda_{12} + 4\lambda_4\lambda_8 & 14\lambda_{14} + 6\lambda_4\lambda_{10} & 8\lambda_4\lambda_{12} & 10\lambda_4\lambda_{14} \\ 10\lambda_{10} & 12\lambda_{12} & 14\lambda_{14} + 6\lambda_4\lambda_{10} & 4\lambda_6\lambda_{10} + 8\lambda_4\lambda_{12} & 6\lambda_6\lambda_{12} + 10\lambda_4\lambda_{14} & 8\lambda_6\lambda_{14} \\ 12\lambda_{12} & 14\lambda_{14} & 8\lambda_4\lambda_{12} & 6\lambda_6\lambda_{12} + 10\lambda_4\lambda_{14} & 4\lambda_8\lambda_{12} + 8\lambda_6\lambda_{14} & 6\lambda_8\lambda_{14} \\ 14\lambda_{14} & 0 & 10\lambda_4\lambda_{14} & 8\lambda_6\lambda_{14} & 6\lambda_8\lambda_{14} & 4\lambda_{10}\lambda_{14} \end{pmatrix} - \frac{1}{7} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 20\lambda_4^2 & 16\lambda_4\lambda_6 & 12\lambda_4\lambda_8 & 8\lambda_4\lambda_{10} & 4\lambda_4\lambda_{12} \\ 0 & 16\lambda_4\lambda_6 & 24\lambda_6^2 & 18\lambda_6\lambda_8 & 12\lambda_6\lambda_{10} & 6\lambda_6\lambda_{12} \\ 0 & 12\lambda_4\lambda_8 & 18\lambda_6\lambda_8 & 24\lambda_8^2 & 16\lambda_8\lambda_{10} & 8\lambda_8\lambda_{12} \\ 0 & 8\lambda_4\lambda_{10} & 12\lambda_6\lambda_{10} & 16\lambda_8\lambda_{10} & 20\lambda_{10}^2 & 10\lambda_{10}\lambda_{12} \\ 0 & 4\lambda_4\lambda_{12} & 6\lambda_6\lambda_{12} & 8\lambda_8\lambda_{12} & 10\lambda_{10}\lambda_{12} & 12\lambda_{12}^2 \end{pmatrix}. \quad (6)$$

A straightforward calculation gives $\det T = -\frac{64}{7}R(\lambda)$ and $(R(\lambda) = 0) \Leftrightarrow (\det T = 0)$, therefore the vector fields L_{2k} are independent at any point of $\mathcal{B} = \mathbb{C}^6 \setminus \Sigma$. We have

$$(L_0, L_2, L_4, L_6, L_8, L_{10}) \det T = (84, 0, 40\lambda_4, 24\lambda_6, 12\lambda_8, 4\lambda_{10}) \det T,$$

so the vector fields L_{2k} are tangent to Σ . \square

Lemma 4.3. *For $g = 3$ we have*

$$\begin{pmatrix} [L_2, L_4] \\ [L_2, L_6] \\ [L_2, L_8] \\ [L_2, L_{10}] \\ [L_4, L_6] \\ [L_4, L_8] \\ [L_4, L_{10}] \\ [L_6, L_8] \\ [L_6, L_{10}] \\ [L_8, L_{10}] \end{pmatrix} = \mathcal{M} \begin{pmatrix} L_0 \\ L_2 \\ L_4 \\ L_6 \\ L_8 \\ L_{10} \end{pmatrix} \text{ for } \mathcal{M} = \frac{2}{7} \begin{pmatrix} 8\lambda_6 & -8\lambda_4 & 0 & 7 & 0 & 0 \\ 6\lambda_8 & 0 & -6\lambda_4 & 0 & 14 & 0 \\ 4\lambda_{10} & 0 & 0 & -4\lambda_4 & 0 & 21 \\ 2\lambda_{12} & 0 & 0 & 0 & -2\lambda_4 & 0 \\ -7\lambda_{10} & 9\lambda_8 & -9\lambda_6 & 7\lambda_4 & 0 & 7 \\ -14\lambda_{12} & 6\lambda_{10} & 0 & -6\lambda_6 & 14\lambda_4 & 0 \\ -21\lambda_{14} & 3\lambda_{12} & 0 & 0 & -3\lambda_6 & 21\lambda_4 \\ -7\lambda_{14} & -7\lambda_{12} & 8\lambda_{10} & -8\lambda_8 & 7\lambda_6 & 7\lambda_4 \\ 0 & -14\lambda_{14} & 4\lambda_{12} & 0 & -4\lambda_8 & 14\lambda_6 \\ 0 & 0 & -7\lambda_{14} & 5\lambda_{12} & -5\lambda_{10} & 7\lambda_8 \end{pmatrix}. \quad (7)$$

Proof. The proof is a straightforward calculation in polynomial vector fields (3). \square

5. POLYNOMIAL MAP

Consider the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\varphi} & \mathbb{C}^{3g} \\ \downarrow \pi & & \downarrow p \\ \mathcal{B} & \hookrightarrow & \mathbb{C}^{2g} \end{array} \quad (8)$$

Here $\pi : \mathcal{U} \rightarrow \mathcal{B}$ is the bundle described above, $\mathcal{B} \subset \mathbb{C}^{2g}$ is the embedding given by the coordinates λ . The map φ will be given by a set of generators of \mathcal{F} and p will be a polynomial map. We will now describe φ and p .

We use a fundamental result from the theory of hyperelliptic Abelian functions (see [9], Chapter 5): Any hyperelliptic function can be represented as a rational function in $\wp_{1;k}$ and $\wp_{2;k}$, where $k \in \{1, 3, \dots, 2g-1\}$. The following theorem from [4] gives a set of relations between the derivatives of these functions. We use it to determine a set of generators in \mathcal{F} .

Theorem 5.1 ([4]). *For $i, k \in \{1, 3, \dots, 2g-1\}$ we have the relations*

$$\wp_{3;i} = 6\wp_{2;\wp_{1;i}} + 6\wp_{1;i+2} - 2\wp_{0;3,i} + 2\lambda_4\delta_{i,1}, \quad (9)$$

$$\begin{aligned} \wp_{2;i}\wp_{2;k} = & 4(\wp_{2;\wp_{1;i}}\wp_{1;k} + \wp_{1;k}\wp_{1;i+2} + \wp_{1;i}\wp_{1;k+2} + \wp_{0;k+2,i+2}) - \\ & - 2(\wp_{1;i}\wp_{0;3,k} + \wp_{1;k}\wp_{0;3,i} + \wp_{0;k,i+4} + \wp_{0;i,k+4}) + \\ & + 2\lambda_4(\delta_{i,1}\wp_{1;k} + \delta_{k,1}\wp_{1;i}) + 2\lambda_{i+k+4}(2\delta_{i,k} + \delta_{k,i-2} + \delta_{i,k-2}). \end{aligned} \quad (10)$$

Proof. In [4] we have formulas (4.1) and (4.8). Using the notation (2) we get (9) from (4.1) and (10) from (4.8). \square

Corollary 5.2. Consider the map $\varphi : \mathcal{U} \dashrightarrow \mathbb{C}^{\frac{g(g+9)}{2}}$, where in $\mathbb{C}^{\frac{g(g+9)}{2}}$ we denote the coordinates by (x, w, λ) . Here for $x = (x_{i,j}) \in \mathbb{C}^{3g}$ we have $i \in \{1, 2, 3\}$, $j \in \{1, 3, \dots, 2g-1\}$, for $w = (w_{k,l}) \in \mathbb{C}^{\frac{g(g-1)}{2}}$ we have $k, l \in \{3, 5, \dots, 2g-1\}$, $k \leq l$, and for $\lambda = (\lambda_s) \in \mathbb{C}^{2g}$ we have $s \in \{4, 6, \dots, 4g, 4g+2\}$. For $(u, \lambda) = (u_1, u_3, \dots, u_{2g-1}, \lambda_4, \lambda_6, \dots, \lambda_{4g+2})$ set

$$\varphi : (u, \lambda) \mapsto (x_{i,j}, w_{k,l}, \lambda_s) = (\wp_{i,j}(u, \lambda), \wp_{0;k,l}(u, \lambda), \lambda_s).$$

We also denote $x_{i+1} = x_{i,1}$, $i = 1, 2, 3$, and $w_{l,k} = w_{k,l}$.

Then the image of φ lies in $\mathcal{S} \subset \mathbb{C}^{\frac{g(g+9)}{2}}$, where \mathcal{S} is determined by the set of $\frac{g(g+3)}{2}$ equations

$$x_4 = 6x_2^2 + 4x_{1,3} + 2\lambda_4, \quad (11)$$

$$x_{3,k} = 6x_2x_{1,k} + 6x_{1,k+2} - 2w_{3,k}, \quad (12)$$

$$x_3^2 = 4x_2^3 + 4x_2x_{1,3} - 4x_{1,5} + 4w_{3,3} + 4\lambda_4x_2 + 4\lambda_6, \quad (13)$$

$$x_3x_{2,k} = 4x_2^2x_{1,k} + 2x_{1,3}x_{1,k} + 4x_2x_{1,k+2} - 2x_{1,k+4} + \quad (14)$$

$$- 2x_2w_{3,k} + 4w_{3,k+2} - 2w_{5,k} + 2\lambda_4x_{1,k} + 2\lambda_8\delta_{3,k},$$

$$x_{2,j}x_{2,k} = 4x_2x_{1,j}x_{1,k} + 4x_{1,k}x_{1,j+2} + 4x_{1,j}x_{1,k+2} + 4w_{k+2,j+2} - \quad (15)$$

$$- 2x_{1,j}w_{3,k} - 2x_{1,k}w_{3,j} - 2w_{k,j+4} - 2w_{j,k+4} + 2\lambda_{j+k+4}(2\delta_{j,k} + \delta_{k,j-2} + \delta_{j,k-2}),$$

for $j, k \in \{3, \dots, 2g-1\}$ and any variable equal to zero if the index is out of range.

Theorem 5.3. The projection $\pi_1 : \mathbb{C}^{\frac{g(g+9)}{2}} \rightarrow \mathbb{C}^{3g}$ on the first $3g$ coordinates gives the isomorphism $\mathcal{S} \simeq \mathbb{C}^{3g}$. Therefore, the coordinates x uniformize \mathcal{S} .

Corollary 5.4. The projection $\pi_2 : \mathbb{C}^{\frac{g(g+9)}{2}} \rightarrow \mathbb{C}^{\frac{g(g-1)}{2}}$ on the second $\frac{g(g-1)}{2}$ coordinates gives a polynomial map $\mathbb{C}^{3g} \rightarrow \mathbb{C}^{\frac{g(g-1)}{2}}$.

Corollary 5.5. The projection $\pi_3 : \mathbb{C}^{\frac{g(g+9)}{2}} \rightarrow \mathbb{C}^{2g}$ on the last $2g$ coordinates gives a polynomial map $p : \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$.

We obtain the diagram:

$$\begin{array}{ccc} & \mathbb{C}^{\frac{g(g+9)}{2}} & \xlongequal{\quad} \mathbb{C}^{3g} \times \mathbb{C}^{\frac{g(g-1)}{2}} \times \mathbb{C}^{2g} \\ & \uparrow & \swarrow \pi_1 \\ \mathcal{U} & \xrightarrow{\varphi} \mathcal{S} \simeq \mathbb{C}^{3g} & \\ \downarrow \pi & \downarrow p & \swarrow \pi_3 \\ \mathcal{B} & \longrightarrow \mathbb{C}^{2g} & \end{array}$$

Proof of theorem 5.3. Equations (11)–(13) give

$$\lambda_4 = \frac{1}{2}x_4 - 3x_2^2 - 2x_{1,3}, \quad \lambda_6 = \frac{1}{2}x_{3,3} - 2x_{1,5} + \frac{1}{4}x_3^2 + 2x_2^3 - 2x_2x_{1,3} - \frac{1}{2}x_2x_4, \quad (16)$$

$$w_{3,k} = -\frac{1}{2}x_{3,k} + 3x_2x_{1,k} + 3x_{1,k+2}. \quad (17)$$

For $k = 3$ equation (14) with (16) gives

$$\lambda_8 = -\frac{1}{2}(x_2x_{3,3} - x_3x_{2,3} + x_4x_{1,3}) + (4x_2^2 + x_{1,3})x_{1,3} + \frac{1}{2}x_{3,5} - 2(x_2x_{1,5} + x_{1,7}).$$

For $k \geq 5$ equation (14) with (16) gives

$$w_{5,k} = \frac{1}{2}(x_2x_{3,k} - x_3x_{2,k} + x_4x_{1,k}) - (4x_2^2 + x_{1,3})x_{1,k} - x_{3,k+2} + 5(x_2x_{1,k+2} + x_{1,k+4}). \quad (18)$$

Now for equation (15) we can assume $j \geq k$. For $j = k$, $j = k + 2$ and $j \geq k + 2$ respectively we obtain the equations

$$x_{2,k}^2 = 4x_{2,k}x_{1,k}^2 + 8x_{1,k}x_{1,k+2} - 4x_{1,k}w_{3,k} + 4w_{k+2,k+2} - 4w_{k,k+4} + 4\lambda_{2k+4}, \quad (19)$$

$$x_{2,k+2}x_{2,k} = 4(x_{2,k+2}x_{1,k} + x_{1,k}x_{1,k+4} + x_{1,k+2}x_{1,k+2}) - 2(x_{1,k+2}w_{3,k} + x_{1,k}w_{3,k+2}) + 4w_{k+2,k+4} - 2w_{k,k+6} - 2w_{k+2,k+4} + 2\lambda_{2k+6}, \quad (20)$$

$$x_{2,j}x_{2,k} = 4(x_{2,j}x_{1,k} + x_{1,k}x_{1,j+2} + x_{1,j}x_{1,k+2}) - 2(x_{1,j}w_{3,k} + x_{1,k}w_{3,j}) + 4w_{k+2,j+2} - 2w_{k,j+4} - 2w_{j,k+4}. \quad (21)$$

Now (17), (18) and (21) for $k = 3, 5, 7, \dots$ give expressions for $w_{j,k+4}$ in terms of x while (19) and (20) give expressions for λ_{2k+4} and λ_{2k+6} in terms of x and w . We see that the expressions for w and λ are polynomial in x , which proves also Corollaries 5.4 and 5.5. \square

6. GENERATORS OF THE POLYNOMIAL LIA ALGEBRA IN \mathbb{C}^{3g}

Now let us return to Problem 1.1. The \mathcal{F} -module $\text{Der } \mathcal{F}$ is determined by its action on the generators of \mathcal{F} . We take the $3g$ functions $\wp_{i,j}(u, \lambda)$, where $i \in \{1, 2, 3\}$ and $j \in \{1, 3, \dots, 2g-1\}$, as these generators. The values of these functions determine λ by Theorem 5.3.

Denote the ring of polynomials in $\lambda \in \mathbb{C}^{2g}$ by \mathcal{P} . Let us consider the polynomial map $p: \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$. A vector field \mathcal{L} in \mathbb{C}^{3g} will be called projectable for p if there exists a vector field L in \mathbb{C}^{2g} such that

$$\mathcal{L}(p^*f) = p^*L(f) \quad \text{for any } f \in \mathcal{P}.$$

The vector field L will be called the pushforward of \mathcal{L} . A corollary of this definition is that for a projectable vector field \mathcal{L} we have $\mathcal{L}(p^*\mathcal{P}) \subset p^*\mathcal{P}$.

Now we will consider the following problem.

Problem 6.1. *Find $3g$ polynomial vector fields in \mathbb{C}^{3g} projectable for $p: \mathbb{C}^{3g} \rightarrow \mathbb{C}^{2g}$ and independent at any point in $p^{-1}(\mathcal{B})$. Construct their polynomial Lie algebra.*

Let us explain the connection to Problem 1.1. Given a solution to Problem 6.1 for each of the $3g$ vector fields \mathcal{L}_k with pushforwards L_k we will restore the vector fields \mathcal{L}_k projectable for π with pushforwards L_k and such that $\mathcal{L}_k(\varphi^*x_{i,j}) = \varphi^*\mathcal{L}_k(x_{i,j})$ for the coordinate functions $x_{i,j}$ in \mathbb{C}^{3g} . As $\varphi^*x_{i,j}$ are the generators of \mathcal{F} and $\mathcal{L}_k(x_{i,j})$ is a polynomial in $x_{i,j}$, this gives $\mathcal{L}_k(\varphi^*x_{i,j}) \in \mathcal{F}$ and $\mathcal{L}_k \in \text{Der } \mathcal{F}$.

Now we will give a solution to Problem 6.1 for $g = 1, 2, 3$.

We denote by \mathcal{T} the $3g \times 3g$ matrix with elements $\mathcal{L}_k(x_{i,j})$ for $k = \{0, 2, \dots, 4g-2\} \cup \{1, 3, \dots, 2g-1\}$ and $i \in \{1, 2, 3\}$, $j \in \{1, 3, \dots, 2g-1\}$.

The vector field \mathcal{L}_0 is the Euler vector field on \mathbb{C}^{3g} , we have

$$\mathcal{L}_0 = \sum_j (j+1)x_{1,j} \frac{\partial}{\partial x_{1,j}} + (j+2)x_{2,j} \frac{\partial}{\partial x_{2,j}} + (j+3)x_{3,j} \frac{\partial}{\partial x_{3,j}}. \quad (22)$$

Then $\mathcal{L}_0(p^*f) = p^*L_0(f)$ for any $f \in \mathcal{P}$.

Denote by \mathcal{L}_k for $k \in \{1, 3, \dots, 2g-1\}$ polynomial vector fields in \mathbb{C}^{3g} such that $\mathcal{L}_k(\varphi^*(x_{i,j})) = \varphi^*\mathcal{L}_k(x_{i,j})$. We have $\mathcal{L}_k(p^*f) = 0$ for any $f \in \mathcal{P}$.

Lemma 6.2. *We have*

$$\mathcal{L}_1 = \sum_j x_{2,j} \frac{\partial}{\partial x_{1,j}} + x_{3,j} \frac{\partial}{\partial x_{2,j}} + 4(2x_{2,j}x_{2,j} + x_{3,j}x_{1,j} + x_{2,j+2}) \frac{\partial}{\partial x_{3,j}} \quad (23)$$

where $x_{2,2g+1} = 0$.

Proof. By definition we have $\mathcal{L}_1(x_{1,k}) = x_{2,k}$ and $\mathcal{L}_1(x_{2,k}) = x_{3,k}$. Denote the polynomial $\mathcal{L}_1(x_{3,k})$ by $x_{4,k}$. We have $\varphi^*x_{4,k} = \wp_{4;k}$. By definition we have $\mathcal{L}_k(x_2) = x_{2,k}$, $\mathcal{L}_k(x_3) = x_{3,k}$, $\mathcal{L}_k(x_4) = x_{4,k}$. Denote the polynomial $\mathcal{L}_k(x_{1,3})$ by $p_{1,3,k}$. Therefore, we have $\varphi^*p_{1,3,k} = \wp_{1;3,k}$ and $\mathcal{L}_1(w_{3,k}) = p_{1,3,k}$.

Applying \mathcal{L}_k and \mathcal{L}_1 to (11) and (12) we get

$$x_{4,k} = 12x_2x_{2,k} + 4p_{1,3,k}, \quad x_{4,k} = 6x_3x_{1,k} + 6x_2x_{2,k} + 6x_{2,k+2} - 2p_{1,3,k},$$

thus $x_{4,k} = 8x_2x_{2,k} + 4x_3x_{1,k} + 4x_{2,k+2}$. \square

Lemma 6.3. *For $s = 3, 5, \dots, 2g - 1$ we have*

$$\begin{aligned} \mathcal{L}_s &= x_{2,s} \frac{\partial}{\partial x_2} + x_{3,s} \frac{\partial}{\partial x_3} + \mathcal{L}_1(x_{3,s}) \frac{\partial}{\partial x_4} + \\ &+ \sum_{k=1}^{g-1} \mathcal{L}_1(w_{s,2k+1}) \frac{\partial}{\partial x_{1,2k+1}} + \mathcal{L}_1(\mathcal{L}_1(w_{s,2k+1})) \frac{\partial}{\partial x_{2,2k+1}} + \mathcal{L}_1(\mathcal{L}_1(\mathcal{L}_1(w_{s,2k+1}))) \frac{\partial}{\partial x_{3,2k+1}}. \end{aligned} \quad (24)$$

Proof. By definition the vector fields \mathcal{L}_1 and \mathcal{L}_s for $s = 3, 5, \dots, 2g - 1$ commute. We have $\mathcal{L}_s(x_i) = x_{i,s}$ and $\mathcal{L}_s(x_{1,2k+1}) = \mathcal{L}_1(w_{s,2k+1})$. This determines the coefficients. \square

Corollary 6.4. *Knowing \mathcal{L}_1 and $\mathcal{L}_s(x_{1,j})$ determines the coefficients $\mathcal{L}_s(x_{2,j})$ and $\mathcal{L}_s(x_{3,j})$.*

7. GENUS 1

In this section we give polynomial vector fields satisfying Problem 6.1 in the case $g = 1$. The map p takes the form

$$\lambda_4 = -3x_2^2 + \frac{1}{2}x_4, \quad \lambda_6 = 2x_2^3 + \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_4.$$

The vector fields are

$$\begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix} = \begin{pmatrix} 2x_2 & 3x_3 & 4x_4 \\ x_3 & x_4 & 12x_2x_3 \\ \frac{2}{3}x_4 - 2x_2^2 & 3x_2x_3 & 2x_2x_4 + 3x_3^2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_4} \end{pmatrix}.$$

The middle matrix is \mathcal{T} . We have $\det \mathcal{T} = 4 \det T$ for T determined by (4). Therefore, the vector fields \mathcal{L}_k are independent at any point of $p^{-1}(\mathcal{B})$.

The polynomial Lie algebra is

$$[\mathcal{L}_0, \mathcal{L}_1] = \mathcal{L}_1, \quad [\mathcal{L}_0, \mathcal{L}_2] = 2\mathcal{L}_2, \quad [\mathcal{L}_1, \mathcal{L}_2] = x_2\mathcal{L}_1.$$

8. GENUS 2

In this section we give polynomial vector fields satisfying Problem 6.1 in the case $g = 2$.

In the formulas below we write y_4 instead of $x_{1,3}$, y_5 instead of $x_{2,3}$ and y_6 instead of $x_{3,3}$ to shorten down the formulas.

The map p takes the form

$$\begin{aligned} \lambda_4 &= -3x_2^2 + \frac{1}{2}x_4 - 2y_4, & \lambda_6 &= 2x_2^3 + \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_4 - 2x_2y_4 + \frac{1}{2}y_6, \\ \lambda_8 &= (4x_2^2 + y_4)y_4 - \frac{1}{2}(x_4y_4 - x_3y_5 + x_2y_6), & \lambda_{10} &= 2x_2y_4^2 + \frac{1}{4}y_5^2 - \frac{1}{2}y_4y_6. \end{aligned} \quad (25)$$

By (22) and (23) we have

$$\begin{aligned}\mathcal{L}_0 &= 2x_2 \frac{\partial}{\partial x_2} + 3x_3 \frac{\partial}{\partial x_3} + 4x_4 \frac{\partial}{\partial x_4} + 4y_4 \frac{\partial}{\partial y_4} + 5y_5 \frac{\partial}{\partial y_5} + 6y_6 \frac{\partial}{\partial y_6}, \\ \mathcal{L}_1 &= x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + 4(3x_2x_3 + y_5) \frac{\partial}{\partial x_4} + y_5 \frac{\partial}{\partial y_4} + y_6 \frac{\partial}{\partial y_5} + 4(2x_2y_5 + x_3y_4) \frac{\partial}{\partial y_6}.\end{aligned}$$

According to (24) and Corollary 6.4 the vector field \mathcal{L}_3 is determined by

$$\mathcal{L}_3(x_2) = y_5, \quad \mathcal{L}_3(y_4) = x_3y_4 - x_2y_5.$$

We have

$$\begin{aligned}\mathcal{L}_3(x_3) &= y_6, & \mathcal{L}_3(y_5) &= x_4y_4 - x_2y_6, \\ \mathcal{L}_3(x_4) &= 4(2x_2y_5 + x_3y_4), & \mathcal{L}_3(y_6) &= 8x_2x_3y_4 - 8x_2^2y_5 + x_4y_5 - x_3y_6 + 4y_4y_5.\end{aligned}$$

Set $p_7 = \mathcal{L}_1(w_6) = \mathcal{L}_3(y_4)$, $w_9 = \mathcal{L}_3(w_6)$. By (17) we have $w_6 = 3x_2y_4 - \frac{1}{2}y_6$,

$$p_7 = x_3y_4 - x_2y_5, \quad w_9 = -x_2x_3y_4 + x_2^2y_5 - \frac{1}{2}(x_4y_5 - x_3y_6) + y_4y_5.$$

Now we introduce the vector fields $\widehat{\mathcal{L}}_2, \widehat{\mathcal{L}}_4, \widehat{\mathcal{L}}_6$. They are determined up to constants α, β, γ_1 and γ_2 by $\widehat{\mathcal{L}}_2 = \mathcal{L}_2$, $\widehat{\mathcal{L}}_4 = \mathcal{L}_4 + \alpha x_3 \mathcal{L}_1$ and $\widehat{\mathcal{L}}_6 = \mathcal{L}_6 + \beta x_3 \mathcal{L}_3 + (\gamma_1 y_5 + \gamma_2 x_2 x_3) \mathcal{L}_1$, where

$$\begin{aligned}\mathcal{L}_2(x_2) &= \frac{8}{5}\lambda_4 + 2x_2^2 + 4y_4, & \mathcal{L}_2(y_4) &= -\frac{4}{5}\lambda_4x_2 + 2x_2y_4, \\ \mathcal{L}_2(x_3) &= 3x_2x_3 + 5y_5, & \mathcal{L}_2(y_5) &= -\frac{4}{5}\lambda_4x_3 + 3x_3y_4, \\ \mathcal{L}_2(x_4) &= 2x_2x_4 + 3x_3^2 + 6y_6, & \mathcal{L}_2(y_6) &= -\frac{4}{5}\lambda_4x_4 + 4x_4y_4 + 3x_3y_5 - 2x_2y_6, \\ \mathcal{L}_4(x_2) &= \frac{2}{5}\lambda_6 - 2x_2y_4 + y_6, & \mathcal{L}_4(x_3) &= x_3y_4 + 5x_2y_5, & \mathcal{L}_4(x_4) &= 6x_3y_5 + 4x_2y_6, \\ \mathcal{L}_4(y_4) &= -\frac{6}{5}\lambda_6x_2 + 2\lambda_4y_4 - 4x_2^2y_4 + x_4y_4 - \frac{1}{2}x_3y_5, \\ \mathcal{L}_4(y_5) &= -\frac{6}{5}\lambda_6x_3 + 2\lambda_4y_5 + 2x_2x_3y_4 - 2x_2^2y_5 + 4y_4y_5 - w_9, \\ \mathcal{L}_4(y_6) &= -\frac{6}{5}\lambda_6x_4 + 2\lambda_4y_6 + x_3^2y_4 + 2x_2x_4y_4 - x_2x_3y_5 - 2x_2^2y_6 + 5y_5^2 + 2y_4y_6, \\ \mathcal{L}_6(x_2) &= \frac{1}{5}\lambda_8 + \frac{1}{2}(x_4y_4 - x_2y_6) - y_4^2, & \mathcal{L}_6(x_3) &= 3x_2p_7 - w_9, \\ \mathcal{L}_6(x_4) &= 2x_3^2y_4 + 4x_2x_4y_4 - 2x_2x_3y_5 - 4x_2^2y_6 + y_5^2 - 2y_4y_6, \\ \mathcal{L}_6(y_4) &= -\frac{8}{5}\lambda_8x_2 + 2\lambda_6y_4 - 2x_2y_4^2 - y_5^2 + y_4y_6, \\ \mathcal{L}_6(y_5) &= -\frac{8}{5}\lambda_8x_3 + 2\lambda_6y_5 + x_3y_4^2 + 5x_2y_4y_5 - y_5y_6, \\ \mathcal{L}_6(y_6) &= -\frac{8}{5}\lambda_8x_4 + 2\lambda_6y_6 + 3x_3y_4y_5 - 3x_2y_5^2 + 6x_2y_4y_6 - y_6^2.\end{aligned}$$

Lemma 8.1. *The vector fields \mathcal{L}_k , $k = 0, 1, 3$, and $\widehat{\mathcal{L}}_k$, $k = 2, 4, 6$, give a solution to Problem 6.1.*

Proof. We have $\mathcal{L}_1(p^*(f)) = \mathcal{L}_3(p^*(f)) = 0$ for any $f \in \mathcal{P}$. So for the proof it is sufficient to check the condition $\widehat{\mathcal{L}}_k(p^*(f)) = p^*(L_k(f))$ for the generators $f = \lambda_4, \lambda_6, \lambda_8, \lambda_{10}$ of \mathcal{P} in the case $\alpha = \beta = \gamma_1 = \gamma_2 = 0$. This is a straightforward calculation using the explicit

polynomial vector fields $\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6$ and L_2, L_4, L_6 (see Section 4) and their action on the polynomials (25).

We have $\det \mathcal{T} = -16 \det T$ for T determined by (5). Therefore, the vector fields $\widehat{\mathcal{L}}_k$ are independent at any point of $p^{-1}(\mathcal{B})$. \square

The polynomial Lie algebra is

$$\begin{aligned}
[\mathcal{L}_0, \mathcal{L}_k] &= k\mathcal{L}_k, & [\mathcal{L}_0, \widehat{\mathcal{L}}_k] &= k\widehat{\mathcal{L}}_k, & [\mathcal{L}_1, \mathcal{L}_3] &= 0, \\
[\mathcal{L}_1, \widehat{\mathcal{L}}_2] &= x_2\mathcal{L}_1 - \mathcal{L}_3, & [\mathcal{L}_1, \widehat{\mathcal{L}}_4] &= y_4\mathcal{L}_1 + x_2\mathcal{L}_3 + \alpha x_4\mathcal{L}_1, \\
[\mathcal{L}_1, \widehat{\mathcal{L}}_6] &= y_4\mathcal{L}_3 + (\gamma_2(x_3^2 + x_2x_4) + \gamma_1y_6)\mathcal{L}_1 + \beta x_4\mathcal{L}_3, \\
[\mathcal{L}_3, \widehat{\mathcal{L}}_2] &= \left(y_4 + \frac{4}{5}\lambda_4\right)\mathcal{L}_1, & [\mathcal{L}_3, \widehat{\mathcal{L}}_4] &= \left(w_6 + \frac{6}{5}\lambda_6\right)\mathcal{L}_1 + (y_4 - \lambda_4)\mathcal{L}_3 + \alpha y_6\mathcal{L}_1, \\
[\mathcal{L}_3, \widehat{\mathcal{L}}_6] &= \frac{3}{5}\lambda_8\mathcal{L}_1 + w_6\mathcal{L}_3 + (\gamma_1x_4y_4 + \gamma_2x_3y_5 - (\gamma_1 - \gamma_2)x_2y_6)\mathcal{L}_1 + \beta y_6\mathcal{L}_3, \\
[\widehat{\mathcal{L}}_2, \widehat{\mathcal{L}}_4] &= \frac{8}{5}\lambda_6\mathcal{L}_0 - \frac{8}{5}\lambda_4\widehat{\mathcal{L}}_2 + 2\widehat{\mathcal{L}}_6 - \frac{1}{2}y_5\mathcal{L}_1 + \frac{1}{2}x_3\mathcal{L}_3 + \\
&\quad + (2(\alpha - \gamma_2)x_2x_3 + (5\alpha - 2\gamma_1)y_5)\mathcal{L}_1 + (\alpha - 2\beta)x_3\mathcal{L}_3, \\
[\widehat{\mathcal{L}}_2, \widehat{\mathcal{L}}_6] &= \frac{4}{5}\lambda_8\mathcal{L}_0 - \frac{4}{5}\lambda_4\widehat{\mathcal{L}}_4 - \frac{1}{2}p_7\mathcal{L}_1 + \frac{1}{2}y_5\mathcal{L}_3 + \\
&\quad + \frac{1}{5}(2(\alpha - \beta - \gamma_1)(x_4 - 6x_2^2) + 4\gamma_2(x_4 - x_2^2 + y_4) - (8\alpha - 3\beta - 23\gamma_1)y_4)x_3\mathcal{L}_1 - \\
&\quad - (\gamma_1 - 5\gamma_2)x_2y_5\mathcal{L}_1 + ((5\beta + \gamma_1)y_5 + (3\beta + \gamma_2)x_2x_3)\mathcal{L}_3, \\
[\widehat{\mathcal{L}}_4, \widehat{\mathcal{L}}_6] &= -2\lambda_{10}\mathcal{L}_0 + \frac{6}{5}\lambda_8\widehat{\mathcal{L}}_2 - \frac{6}{5}\lambda_6\widehat{\mathcal{L}}_4 + 2\lambda_4\widehat{\mathcal{L}}_6 - \frac{1}{2}w_9\mathcal{L}_1 + \frac{1}{2}p_7\mathcal{L}_3 + \\
&\quad + (\alpha - \beta - \gamma_1 + 2\gamma_2)\left(\frac{6}{5}\lambda_6 + w_6 + y_6\right)x_3\mathcal{L}_1 + \\
&\quad + \left(2\gamma_2x_2^3 - \frac{1}{2}\gamma_2x_3^2 + (6\gamma_1 - 7\alpha)x_2y_4 + (\beta - \gamma_2)y_6\right)x_3\mathcal{L}_1 + \\
&\quad + \left((4\alpha - 3\gamma_1 + 5\gamma_2)x_2^2 - \frac{1}{2}(\alpha - \gamma_1)x_4 + (\alpha + 2\gamma_1)y_4\right)y_5\mathcal{L}_1 + \\
&\quad + \alpha(\gamma_2x_3^3 - \gamma_1x_4y_5 - (\beta - \gamma_1)x_3y_6)\mathcal{L}_1 + \\
&\quad + \left((3\beta - \gamma_2)x_2^2x_3 + \frac{1}{2}\beta(2\alpha - 1)x_3x_4 + (\alpha + 2\beta)x_3y_4 + (5\beta - \gamma_1)x_2y_5\right)\mathcal{L}_3.
\end{aligned}$$

Remark 8.2. We give here the full form of the polynomial Lie algebra for arbitrary constants $\alpha, \beta, \gamma_1, \gamma_2$, as for any of these constants the polynomial vector fields satisfy Problem 6.1 in the case $g = 2$ and thus give a solution to Problem 1.1. To specify these constants some complimentary condition is needed. In [11] the condition that the vector fields form a Witt algebra is taken. In the current work to specify one Lie algebra in this space of parameters we use the following condition on the polynomial Lie algebra:

$$\begin{pmatrix} [\mathcal{L}_1, \mathcal{L}_0] \\ [\mathcal{L}_1, \mathcal{L}_2] \\ [\mathcal{L}_1, \mathcal{L}_4] \\ [\mathcal{L}_1, \mathcal{L}_6] \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ x_2 & -1 \\ y_4 & x_2 \\ 0 & y_4 \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \end{pmatrix}.$$

Let us note that this condition leads to the polynomial Lie algebra obtained in [3] and the solution to Problem 1.1 described in Section 3.2.

9. GENUS 3

In this section we give polynomial vector fields satisfying Problem 6.1 in the case $g = 3$.

Analogously to the genus $g = 2$ case, the vector fields are determined ambiguously up to some parameters. These parameters are determined using the condition (26).

In the formulas below we write y_4 instead of $x_{1,3}$, y_5 instead of $x_{2,3}$, y_6 instead of $x_{3,3}$, z_6 instead of $x_{1,5}$, z_7 instead of $x_{2,5}$, z_8 instead of $x_{3,5}$, w_6 instead of $w_{3,3}$, w_8 instead of $x_{3,5}$, and w_{10} instead of $w_{5,5}$ to shorten down the formulas.

The map p takes the form

$$\begin{aligned}\lambda_4 &= -3x_2^2 + \frac{1}{2}x_4 - 2y_4, & \lambda_6 &= 2x_2^3 + \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_4 - 2x_2y_4 + \frac{1}{2}y_6 - 2z_6, \\ \lambda_8 &= 4x_2^2y_4 - \frac{1}{2}(x_4y_4 - x_3y_5 + x_2y_6) + y_4^2 - 2x_2z_6 + \frac{1}{2}z_8, \\ \lambda_{10} &= 2x_2y_4^2 + \frac{1}{4}y_5^2 - \frac{1}{2}y_4y_6 - \frac{1}{2}(x_4z_6 - x_3z_7 + x_2z_8) + (4x_2^2 + 2y_4)z_6, \\ \lambda_{12} &= 4x_2y_4z_6 - \frac{1}{2}(y_6z_6 - y_5z_7 + y_4z_8) + z_6^2, & \lambda_{14} &= 2x_2z_6^2 + \frac{1}{4}z_7^2 - \frac{1}{2}z_6z_8.\end{aligned}$$

By (22) we have

$$\mathcal{L}_0 = \sum_{k=2,3,4} kx_k \frac{\partial}{\partial x_k} + \sum_{k=4,5,6} ky_k \frac{\partial}{\partial y_k} + \sum_{k=6,7,8} kz_k \frac{\partial}{\partial z_k}.$$

It is Euler vector field, and for all k we will have

$$[\mathcal{L}_0, \mathcal{L}_k] = k\mathcal{L}_k.$$

By (23) we have

$$\begin{aligned}\mathcal{L}_1 &= x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + 4(3x_2x_3 + y_5) \frac{\partial}{\partial x_4} + \\ &\quad + y_5 \frac{\partial}{\partial y_4} + y_6 \frac{\partial}{\partial y_5} + 4(x_3y_4 + 2x_2y_5 + z_7) \frac{\partial}{\partial y_6} + \\ &\quad + z_7 \frac{\partial}{\partial z_6} + z_8 \frac{\partial}{\partial z_7} + 4(x_3z_6 + 2x_2z_7) \frac{\partial}{\partial z_8}.\end{aligned}$$

According to definition and Corollary 6.4, we have

$$[\mathcal{L}_1, \mathcal{L}_3] = 0, \quad [\mathcal{L}_1, \mathcal{L}_5] = 0, \quad [\mathcal{L}_3, \mathcal{L}_5] = 0,$$

and the vector fields \mathcal{L}_3 and \mathcal{L}_5 are determined by

$$\begin{aligned}\mathcal{L}_3(x_2) &= y_5, & \mathcal{L}_3(y_4) &= x_3y_4 - x_2y_5 + z_7, & \mathcal{L}_3(z_6) &= x_3z_6 - x_2z_7, \\ \mathcal{L}_5(x_2) &= z_7, & \mathcal{L}_5(y_4) &= x_3z_6 - x_2z_7, & \mathcal{L}_5(z_6) &= y_5z_6 - y_4z_7.\end{aligned}$$

By (17), (18) we have

$$\begin{aligned}w_6 &= 3x_2y_4 - \frac{1}{2}y_6 + 3z_6, & w_8 &= 3x_2z_6 - \frac{1}{2}z_8, \\ w_{10} &= \frac{1}{2}(x_4z_6 - x_3z_7 + x_2z_8) - (4x_2^2 + y_4)z_6.\end{aligned}$$

Denote $p_7 = \mathcal{L}_1(w_6) = \mathcal{L}_3(y_4)$, $p_9 = \mathcal{L}_1(w_8) = \mathcal{L}_3(z_6) = \mathcal{L}_5(y_4)$, $p_{11} = \mathcal{L}_1(w_{10}) = \mathcal{L}_5(z_6)$, $w_9 = \mathcal{L}_3(w_6)$, $w_{11} = \mathcal{L}_5(w_6) = \mathcal{L}_3(w_8)$, $w_{13} = \mathcal{L}_5(w_8) = \mathcal{L}_3(w_{10})$, $w_{15} = \mathcal{L}_5(w_{10})$. We have

$$p_7 = x_3y_4 - x_2y_5 + z_7, \quad p_9 = x_3z_6 - x_2z_7, \quad p_{11} = y_5z_6 - y_4z_7,$$

$$\begin{aligned}
w_9 &= -x_3x_2y_4 + x_2^2y_5 - \frac{1}{2}x_4y_5 + \frac{1}{2}x_3y_6 + y_4y_5 + x_3z_6 - 2x_2z_7, \\
w_{11} &= -x_2x_3z_6 + y_5z_6 + x_2^2z_7 - \frac{1}{2}x_4z_7 + \frac{1}{2}x_3z_8, \\
w_{13} &= -y_5x_2z_6 + x_2y_4z_7 - \frac{1}{2}y_6z_7 + \frac{1}{2}y_5z_8 + z_6z_7, \\
w_{15} &= -y_4y_5z_6 + y_4^2z_7 + x_3z_6^2 - x_2z_6z_7 - \frac{1}{2}z_7z_8 + \frac{1}{2}z_8p_7 - \frac{1}{2}y_6p_9 + \frac{1}{2}x_4p_{11}.
\end{aligned}$$

Now for $\mathcal{L}_0, \mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6, \mathcal{L}_8, \mathcal{L}_{10}$ we give their commutators with \mathcal{L}_1 :

$$\begin{pmatrix} [\mathcal{L}_1, \mathcal{L}_0] \\ [\mathcal{L}_1, \mathcal{L}_2] \\ [\mathcal{L}_1, \mathcal{L}_4] \\ [\mathcal{L}_1, \mathcal{L}_6] \\ [\mathcal{L}_1, \mathcal{L}_8] \\ [\mathcal{L}_1, \mathcal{L}_{10}] \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ x_2 & -1 & 0 \\ y_4 & x_2 & -1 \\ z_6 & y_4 & x_2 \\ 0 & z_6 & y_4 \\ 0 & 0 & z_6 \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix}. \quad (26)$$

As the right hand sides of the commutators have already been defined, these relations determine $\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6, \mathcal{L}_8, \mathcal{L}_{10}$ given their values on x_2, y_4, z_6 .

Let us give these values:

$$\begin{aligned}
\mathcal{L}_2(x_2) &= \frac{12}{7}\lambda_4 + 2x_2^2 + 4y_4, & \mathcal{L}_2(y_4) &= -\frac{8}{7}\lambda_4x_2 + 2x_2y_4 + 6z_6, & \mathcal{L}_2(z_6) &= -\frac{4}{7}\lambda_4y_4 + 2x_2z_6, \\
\mathcal{L}_4(x_2) &= \frac{4}{7}\lambda_6 - 2x_2y_4 + y_6 + 2z_6, & \mathcal{L}_4(z_6) &= -\frac{6}{7}\lambda_6y_4 - 16x_2^2z_6 + 3x_4z_6 - 8y_4z_6 - \frac{1}{2}x_3z_7, \\
\mathcal{L}_4(y_4) &= -\frac{12}{7}\lambda_6x_2 - 10x_2^2y_4 + 2x_4y_4 - 4y_4^2 - \frac{1}{2}x_3y_5 + 2x_2z_6, \\
\mathcal{L}_6(x_2) &= -\frac{4}{7}\lambda_8 + 4x_2^2y_4 - x_2y_6 - 4x_2z_6 + \frac{1}{2}x_3y_5 + 2z_8, \\
\mathcal{L}_6(y_4) &= -\frac{16}{7}\lambda_8x_2 + 2\lambda_6y_4 - 2x_2y_4^2 - y_5^2 + y_4y_6 - 16x_2^2z_6 + 3x_4z_6 - 6y_4z_6 - \frac{1}{2}x_3z_7, \\
\mathcal{L}_6(z_6) &= -\frac{8}{7}\lambda_8y_4 + 4\lambda_6z_6 - 2x_2y_4z_6 + y_6z_6 - y_5z_7 + 2z_6^2, \\
\mathcal{L}_8(x_2) &= \frac{2}{7}\lambda_{10} + x_4z_6 - 2y_4z_6 - x_2z_8, \\
\mathcal{L}_8(y_4) &= -\frac{6}{7}\lambda_{10}x_2 + x_3^2z_6 - x_2x_4z_6 - 18x_2y_4z_6 + 3y_6z_6 - x_2x_3z_7 - \frac{5}{2}y_5z_7 + x_2^2z_8 + 2y_4z_8 - 6z_6^2, \\
\mathcal{L}_8(z_6) &= -\frac{10}{7}\lambda_{10}y_4 - x_4y_4z_6 + \frac{3}{2}x_3y_5z_6 + 2y_4^2z_6 - 20x_2z_6^2 - z_7^2 + 3z_6z_8 + 4w_6x_2z_6 - \frac{1}{2}z_7p_7, \\
\mathcal{L}_{10}(x_2) &= \frac{1}{7}\lambda_{12} + \frac{1}{2}y_6z_6 - \frac{1}{2}y_4z_8 - z_6^2, \\
\mathcal{L}_{10}(y_4) &= -\frac{3}{7}\lambda_{12}x_2 + \frac{1}{2}x_3y_5z_6 - \frac{1}{2}x_2y_6z_6 - \frac{1}{2}x_3y_4z_7 + \frac{1}{2}x_2y_4z_8 - 11x_2z_6^2 - \frac{3}{2}z_7^2 + 3z_6z_8, \\
\mathcal{L}_{10}(z_6) &= \frac{2}{7}\lambda_{12}y_4 - 4x_2y_4^2z_6 + \frac{1}{2}y_5^2z_6 - y_4y_5z_7 + y_4^2z_8 + 8x_2^2z_6^2 + x_2z_7^2 - 2x_2z_6z_8.
\end{aligned}$$

Lemma 9.1. *For the polynomial vector fields $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_6, \mathcal{L}_5, \mathcal{L}_8, \mathcal{L}_{10}$ defined in this section for $k = 4, 6, 8, 10, 12, 14$ we have*

$$\begin{aligned}\mathcal{L}_0(\lambda_k) &= L_0(\lambda_k), & \mathcal{L}_1(\lambda_k) &= 0, & \mathcal{L}_2(\lambda_k) &= L_2(\lambda_k), \\ \mathcal{L}_4(\lambda_k) &= L_4(\lambda_k), & \mathcal{L}_3(\lambda_k) &= 0, & \mathcal{L}_6(\lambda_k) &= L_6(\lambda_k), \\ \mathcal{L}_8(\lambda_k) &= L_8(\lambda_k), & \mathcal{L}_5(\lambda_k) &= 0, & \mathcal{L}_{10}(\lambda_k) &= L_{10}(\lambda_k).\end{aligned}$$

Proof. The proof is a straightforward calculation obtained by applying the polynomial vector fields \mathcal{L}_s to the polynomials λ_k . \square

Lemma 9.2. *The vector fields $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_6, \mathcal{L}_5, \mathcal{L}_8, \mathcal{L}_{10}$ solve Problem 6.1.*

Proof. By Lemma 9.1 the vector fields \mathcal{L}_k are projectable for p .

We have $\det \mathcal{T} = -64 \det T$ for T determined by (6). Therefore, the vector fields \mathcal{L}_k are independent at any point of $p^{-1}(\mathcal{B})$. \square

Now let us describe the polynomial Lie algebra for the vector fields \mathcal{L}_k . The commutators $[\mathcal{L}_0, \mathcal{L}_k]$, $[\mathcal{L}_1, \mathcal{L}_k]$ and $[\mathcal{L}_3, \mathcal{L}_5]$ have been given above.

The commutators of \mathcal{L}_3 with $\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6, \mathcal{L}_8, \mathcal{L}_{10}$ are

$$\begin{pmatrix} [\mathcal{L}_3, \mathcal{L}_2] \\ [\mathcal{L}_3, \mathcal{L}_4] \\ [\mathcal{L}_3, \mathcal{L}_6] \\ [\mathcal{L}_3, \mathcal{L}_8] \\ [\mathcal{L}_3, \mathcal{L}_{10}] \end{pmatrix} = \begin{pmatrix} y_4 - \lambda_4 & 0 & -3 \\ w_6 & y_4 - \lambda_4 & 0 \\ w_8 & w_6 & y_4 - \lambda_4 \\ 0 & w_8 & w_6 \\ 0 & 0 & w_8 \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix} + \frac{3}{7} \begin{pmatrix} 5\lambda_4 \\ 4\lambda_6 \\ 3\lambda_8 \\ 2\lambda_{10} \\ \lambda_{12} \end{pmatrix} \mathcal{L}_1. \quad (27)$$

The commutators of \mathcal{L}_5 with $\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6, \mathcal{L}_8, \mathcal{L}_{10}$ are

$$\begin{pmatrix} [\mathcal{L}_5, \mathcal{L}_2] \\ [\mathcal{L}_5, \mathcal{L}_4] \\ [\mathcal{L}_5, \mathcal{L}_6] \\ [\mathcal{L}_5, \mathcal{L}_8] \\ [\mathcal{L}_5, \mathcal{L}_{10}] \end{pmatrix} = \begin{pmatrix} z_6 & 0 & 0 \\ w_8 & z_6 & 0 \\ w_{10} & w_8 & z_6 \\ -\lambda_{12} & w_{10} & w_8 \\ -2\lambda_{14} & -\lambda_{12} & w_{10} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix} + \frac{2}{7} \begin{pmatrix} 2\lambda_4 \\ 3\lambda_6 \\ 4\lambda_8 \\ 5\lambda_{10} \\ 6\lambda_{12} \end{pmatrix} \mathcal{L}_3 - \begin{pmatrix} 0 \\ 3\lambda_4 \\ 2\lambda_6 \\ \lambda_8 \\ 0 \end{pmatrix} \mathcal{L}_5. \quad (28)$$

The remaining commutators are

$$\begin{pmatrix} [\mathcal{L}_2, \mathcal{L}_4] \\ [\mathcal{L}_2, \mathcal{L}_6] \\ [\mathcal{L}_2, \mathcal{L}_8] \\ [\mathcal{L}_2, \mathcal{L}_{10}] \\ [\mathcal{L}_4, \mathcal{L}_6] \\ [\mathcal{L}_4, \mathcal{L}_8] \\ [\mathcal{L}_4, \mathcal{L}_{10}] \\ [\mathcal{L}_6, \mathcal{L}_8] \\ [\mathcal{L}_6, \mathcal{L}_{10}] \\ [\mathcal{L}_8, \mathcal{L}_{10}] \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_2 \\ \mathcal{L}_4 \\ \mathcal{L}_6 \\ \mathcal{L}_8 \\ \mathcal{L}_{10} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -y_5 & x_3 & 0 \\ -p_7 - z_7 & y_5 & x_3 \\ -2p_9 & z_7 & y_5 \\ -p_{11} & 0 & z_7 \\ -w_9 & p_7 - 2z_7 & 2y_5 \\ -2w_{11} & 0 & 2p_7 \\ -w_{13} & -p_{11} & 2p_9 \\ -2w_{13} & 2p_{11} - w_{11} & w_9 \\ -w_{15} & -w_{13} & w_{11} + p_{11} \\ 0 & -w_{15} & w_{13} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix},$$

where \mathcal{M} is given by (7).

10. REPRESENTATION OF THE GENERATORS IN CLASSIC FORM

Theorem 10.1. *For genus $g = 3$ the generators of the \mathcal{F} -module $\text{Der } \mathcal{F}$ are*

$$\begin{aligned}
\mathcal{L}_1 &= \partial_{u_1}, & \mathcal{L}_3 &= \partial_{u_3}, & \mathcal{L}_5 &= \partial_{u_5}, \\
\mathcal{L}_0 &= L_0 - u_1 \partial_{u_1} - 3u_3 \partial_{u_3} - 5u_5 \partial_{u_5}, \\
\mathcal{L}_2 &= L_2 - \left(\zeta_1 - \frac{8}{7} \lambda_4 u_3 \right) \partial_{u_1} - \left(u_1 - \frac{4}{7} \lambda_4 u_5 \right) \partial_{u_3} - 3u_3 \partial_{u_5}, \\
\mathcal{L}_4 &= L_4 - \left(\zeta_3 - \frac{12}{7} \lambda_6 u_3 \right) \partial_{u_1} - \left(\zeta_1 + \lambda_4 u_3 - \frac{6}{7} \lambda_6 u_5 \right) \partial_{u_3} - (u_1 + 3\lambda_4 u_5) \partial_{u_5}, \\
\mathcal{L}_6 &= L_6 - \left(\zeta_5 - \frac{9}{7} \lambda_8 u_3 \right) \partial_{u_1} - \left(\zeta_3 - \frac{8}{7} \lambda_8 u_5 \right) \partial_{u_3} - (\zeta_1 + \lambda_4 u_3 + 2\lambda_6 u_5) \partial_{u_5}, \\
\mathcal{L}_8 &= L_8 + \left(\frac{6}{7} \lambda_{10} u_3 - \lambda_{12} u_5 \right) \partial_{u_1} - \left(\zeta_5 - \frac{10}{7} \lambda_{10} u_5 \right) \partial_{u_3} - (\zeta_3 + \lambda_8 u_5) \partial_{u_5}, \\
\mathcal{L}_{10} &= L_{10} + \left(\frac{3}{7} \lambda_{12} u_3 - 2\lambda_{14} u_5 \right) \partial_{u_1} + \frac{5}{7} \lambda_{12} u_5 \partial_{u_3} - \zeta_5 \partial_{u_5},
\end{aligned}$$

where the vector fields L_{2k} are given explicitly by (3).

Proof. To obtain $\mathcal{L}_k f \in \mathcal{F}$ for $f \in \mathcal{F}$ we take \mathcal{L}_k such that $\mathcal{L}_k \varphi^* x_{i,j} = \varphi^* \mathcal{L}_k x_{i,j}$ for the coordinate functions $x_{i,j}$ in \mathbb{C}^{3g} .

We have $\mathcal{L}_1 = \partial_{u_1}$, $\mathcal{L}_3 = \partial_{u_3}$, $\mathcal{L}_5 = \partial_{u_5}$, see Section 2. The vector field \mathcal{L}_0 is the Euler vector field and $\mathcal{L}_0 \in \text{Der } \mathcal{F}$ follows from homogeneity properties of hyperelliptic sigma function. The vector fields $\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6, \mathcal{L}_8, \mathcal{L}_{10}$ are determined by the conditions:

- (1) $\mathcal{L}_{2k} = L_{2k} + f_{2k,1}(u, \lambda) \partial_{u_1} + f_{2k,3}(u, \lambda) \partial_{u_3} + f_{2k,5}(u, \lambda) \partial_{u_5}$,
- (2) the fields \mathcal{L}_k satisfy commutation relations obtained by φ^* from (26), (27), (28).

The condition (2) determines the coefficients $f_{2k,j}(u, \lambda)$, $j = 1, 3, 5$, up to constants in u_1, u_3, u_5 . The grading of the variables and the condition $[\mathcal{L}_0, \mathcal{L}_k] = k\mathcal{L}_k$ determines these constants. \square

Corollary 10.2. *The Lie algebra for the generators from Theorem 10.1 of the \mathcal{F} -module $\text{Der } \mathcal{F}$ in the genus $g = 3$ case is*

$$[\mathcal{L}_0, \mathcal{L}_k] = k\mathcal{L}_k, \quad [\mathcal{L}_1, \mathcal{L}_3] = 0, \quad [\mathcal{L}_1, \mathcal{L}_5] = 0, \quad [\mathcal{L}_3, \mathcal{L}_5] = 0,$$

$$\begin{aligned}
\begin{pmatrix} [\mathcal{L}_1, \mathcal{L}_2] \\ [\mathcal{L}_1, \mathcal{L}_4] \\ [\mathcal{L}_1, \mathcal{L}_6] \\ [\mathcal{L}_1, \mathcal{L}_8] \\ [\mathcal{L}_1, \mathcal{L}_{10}] \end{pmatrix} &= \begin{pmatrix} \wp_2 & -1 & 0 \\ \wp_{1;3} & \wp_2 & -1 \\ \wp_{1;5} & \wp_{1;3} & \wp_2 \\ 0 & \wp_{1;5} & \wp_{1;3} \\ 0 & 0 & \wp_{1;5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix}, \\
\begin{pmatrix} [\mathcal{L}_3, \mathcal{L}_2] \\ [\mathcal{L}_3, \mathcal{L}_4] \\ [\mathcal{L}_3, \mathcal{L}_6] \\ [\mathcal{L}_3, \mathcal{L}_8] \\ [\mathcal{L}_3, \mathcal{L}_{10}] \end{pmatrix} &= \begin{pmatrix} \wp_{1;3} - \lambda_4 & 0 & -3 \\ \wp_{0;3,3} & \wp_{1;3} - \lambda_4 & 0 \\ \wp_{0;3,5} & \wp_{0;3,3} & \wp_{1;3} - \lambda_4 \\ 0 & \wp_{0;3,5} & \wp_{0;3,3} \\ 0 & 0 & \wp_{0;3,5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix} + \frac{3}{7} \begin{pmatrix} 5\lambda_4 \\ 4\lambda_6 \\ 3\lambda_8 \\ 2\lambda_{10} \\ \lambda_{12} \end{pmatrix} \mathcal{L}_1, \\
\begin{pmatrix} [\mathcal{L}_5, \mathcal{L}_2] \\ [\mathcal{L}_5, \mathcal{L}_4] \\ [\mathcal{L}_5, \mathcal{L}_6] \\ [\mathcal{L}_5, \mathcal{L}_8] \\ [\mathcal{L}_5, \mathcal{L}_{10}] \end{pmatrix} &= \begin{pmatrix} \wp_{1;5} & 0 & 0 \\ \wp_{0;3,5} & \wp_{1;5} & 0 \\ \wp_{0;5,5} & \wp_{0;3,5} & \wp_{1;5} \\ -\lambda_{12} & \wp_{0;5,5} & \wp_{0;3,5} \\ -2\lambda_{14} & -\lambda_{12} & \wp_{0;5,5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix} + \frac{2}{7} \begin{pmatrix} 2\lambda_4 \\ 3\lambda_6 \\ 4\lambda_8 \\ 5\lambda_{10} \\ 6\lambda_{12} \end{pmatrix} \mathcal{L}_3 - \begin{pmatrix} 0 \\ 3\lambda_4 \\ 2\lambda_6 \\ \lambda_8 \\ 0 \end{pmatrix} \mathcal{L}_5,
\end{aligned}$$

$$\begin{pmatrix} [\mathcal{L}_2, \mathcal{L}_4] \\ [\mathcal{L}_2, \mathcal{L}_6] \\ [\mathcal{L}_2, \mathcal{L}_8] \\ [\mathcal{L}_2, \mathcal{L}_{10}] \\ [\mathcal{L}_4, \mathcal{L}_6] \\ [\mathcal{L}_4, \mathcal{L}_8] \\ [\mathcal{L}_4, \mathcal{L}_{10}] \\ [\mathcal{L}_6, \mathcal{L}_8] \\ [\mathcal{L}_6, \mathcal{L}_{10}] \\ [\mathcal{L}_8, \mathcal{L}_{10}] \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_2 \\ \mathcal{L}_4 \\ \mathcal{L}_6 \\ \mathcal{L}_8 \\ \mathcal{L}_{10} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\wp_{2;3} & \wp_3 & 0 \\ -\wp_{1;3,3} - \wp_{2;5} & \wp_{2;3} & \wp_3 \\ -2\wp_{1;3,5} & \wp_{2;5} & \wp_{2;3} \\ -\wp_{1;5,5} & 0 & \wp_{2;5} \\ -\wp_{0;3,3,3} & \wp_{1;3,3} - 2\wp_{2;5} & 2\wp_{2;3} \\ -2\wp_{0;3,3,5} & 0 & 2\wp_{1;3,3} \\ -\wp_{0;3,5,5} & -\wp_{1;5,5} & 2\wp_{1;3,5} \\ -2\wp_{0;3,5,5} & 2\wp_{1;5,5} - \wp_{0;3,3,5} & \wp_{0;3,3,3} \\ -\wp_{0;5,5,5} & -\wp_{0;3,5,5} & \wp_{0;3,3,5} + \wp_{1;5,5} \\ 0 & -\wp_{0;5,5,5} & \wp_{0;3,5,5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix},$$

where \mathcal{M} is given by (7).

The proof. is obtained by applying φ^* to the polynomial Lie algebra in Section 9. \square

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